A New Derivation of Faulhaber's Formula

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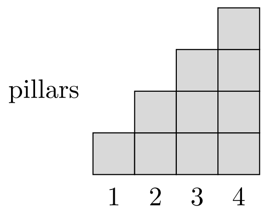
Faulhaber's formula is a closed-form expression which allows one to calculate the sum $\sum_1^n i^k$ for any fixed integers k and n. A new method of deriving Faulhaber's formula is presented which argues that

$$\sum_{i=1}^{n} i^k = n\sum_{i=1}^{n} i^{k-1} - \sum_{m=1}^{n-1}\sum_{i=1}^{m} i^{k-1}$$

Though one can verify this equation through induction, discovering it came via some geometrical considerations which are explained below. Since I do not yet know how to find a closed form expression of the $\sum_{m=1}^{n}\sum_{i=1}^{m} i^{k-1}$ term I cannot use the above expression to derive Faulhaber's formula, though I do not doubt that it is possible to do so.

I was first turned to the problem after considering whether there was a general formula for all those sums of the form $\sum_1^n i^k$ which one tends to be asked to evaluate in introductory calculus texts, when one is introduced to the cumbersome way of evaluating the areas of parabolic segments defined by functions of the form $x^k, k \in \mathbb{N}$. I recalled the story of how the young Gauss solved the problem for $k = 1$ by noticing the symmetry that could be formed by pairing the ith term in $\sum_1^n i^k$ with the $(n+1) - i$th term. This solution has a geometrical flavour, in that you take the n-tuple representing the sum, make a copy of it, flip it about, and voila, the sum $\frac{n(n+1)}{2}$ falls out, and so I wondered, what if the solution to the problem for larger values of k also had a geometrical flavour?

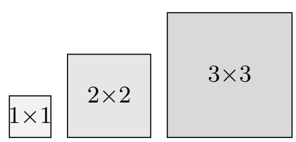
Consider the sum $\sum_1^n i^2$, what visual interpretation could we provide it with? Well, the square of a number n is just a sum of n terms, each term being equal to n. If we were to treat each term in the sum as a "pillar", then the sum would form a triangle of height and length n, looking like so:



We calculate the sum of each term in the triangle, and thereby evaluate $\sum_1^n i^2$, by noticing that the base of the triangle equals $\sum_1^n i$, and that as we move up the triangle's horizontal layers, we remove a term from the sum $\sum_1^n i^2$, such that $\sum_1^n i^2 = n \sum_1^n i - \sum_{m=1}^{n-1} \sum_{i=1}^{m} i$, or put another way, $\sum_1^n i^2$ equals n copies of the sum $\sum_1^n i$, which obviously form a square of height and width n, minus all the partial sums from 1 to $n-1$, from which the triangle appears.

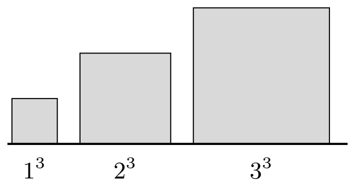
One may express the concern that this is dandy, but that we may run into a problem with higher values of k. This did not end up being the case, and proving the case for $k = 4$ gives us a justification for why it works in general, as we shall see.

We proceed by the same method, asking what visual representation we can ascribe to $\sum_1^n i^3$. The answer is "squares" of numbers, since if i^2 represents a pillar, i^3 represents i copies of that pillar of height i.

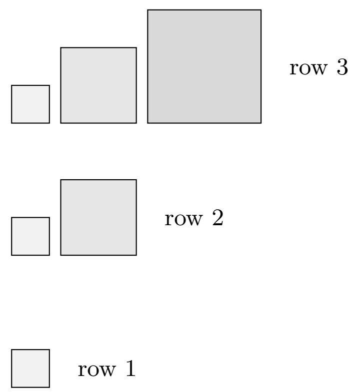


We can line up these squares like a pyramid on its side, and we can proceed to verify that $\sum_{i=1}^n i^3 = n \sum_{i=1}^n i^2 - \sum_{m=1}^{n-1} \sum_{i=1}^{m} i^2$ by noticing that this row of n squares is formed by taking n triangles of the type discussed above, lining them up together to form a prism of height, width, and length n, and then subtracting pillar 1 from the second triangle, pillars 1 and 2 (i.e. $1^2 + 2^2$) from the third, and finally pillars 1 to $n-1$ from triangle n in the prism.

We proceed to the aforementioned final case we have to check. It should be clear now that the visual representation of $\sum_1^n i^4$ consists of a row of cubes (what else would be formed by taking i copies of sheets of height and width i?). If we take our previous representation of $\sum_1^n i^3$ as a row of squares and turn each square 90 degrees in the same direction, we get the following object — a row of cubes drawn in isometric projection, with cube i having side length i:

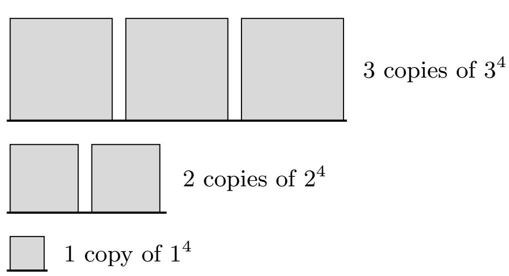


By now it should be clear what's coming. We can see that our row of n cubes is formed by taking what is now a "staircase", formed by lining up n copies of our rotated row of squares, and subtracting square 1 from the second row of squares, square 1 and 2 from the third, etc. all the way until we are left with our row of cubes, thereby demonstrating that $\sum_{i=1}^{n} i^4 = n\sum_{i=1}^{n} i^3 - \sum_{m=1}^{n-1}\sum_{i=1}^{m} i^3$.



(staircase of rows of squares, each row k contains squares $1^2, \ldots, k^2$)

Having demonstrated the truth of the case $k = 4$, we can now generalise by noticing that for $k = 5$, the geometrical interpretation corresponds to new kinds of staircases, where at each ith "step" of the staircase there are now i times as many cubes. So if we imagine a staircase formed from n copies of our cubical staircase (i.e. n copies of the geometrical representation of $\sum_1^n i^4$), and if we subtract the partial terms of $\sum_1^n i^4$ from this new creation, we get the geometric interpretation of $\sum_1^n i^5$:



That this line of reasoning applies to cases $k > 5$ became clear when I saw that the geometrical arguments used to demonstrate cases $k = 2$ to $k = 5$ can be used repeatedly and cyclically (i.e. to demonstrate the validity of $k = 5$ we use the same method which we employed to go from $k = 3$ to $k = 4$, and to demonstrate $k = 6$ we use the same method as we did to demonstrate the veracity of $k = 3$ from $k = 2$, etc.) to demonstrate that the formula holds for all cases $k > 5$, which led me to check whether $\sum_{i=1}^{n} i^k = n \sum_{i=1}^{n} i^{k-1} - \sum_{m=1}^{n-1} \sum_{i=1}^{m} i^{k-1}$ was true by induction.